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SPECIALIST MATHEMATICS
Teach Yourself Series
Topic 11: Vector Calculus

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Vector Calculus

The objective of studying this topic is to differentiate and antidifferentiate a vector function to analyse the motion of a particle along a curve. It includes understanding the concept of position vectors as a function of time and to be able to represent the path of a particle in two and three dimensions (as a vector function). This branch of mathematics helps us to understand motion concepts in Physics.

Vector equations and Position vectors as a function of time

As it appears in Unit 4

- These are used to describe the motion of particles.
- The paths described are the same but the particles are at different locations at a given time.
- Vector equations can be converted to Cartesian equations by substituting the $\underset{\sim}{i}$ component as x and the $\underset{\sim}{j}$ as y, and eliminating r and θ to get a relation between x and y only.
- Position vectors tell us about the path of an object and its motion in time.
- The position vector of a particle at time t in two dimensions is given by

$$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$$

- The parametric form of the path of a particle is given by

$$x = x(t), \quad y = y(t)$$

Eg. A particle moves so that its position vector $\underset{\sim}{r}$ at time t, $t \geq 0$ is $\underset{\sim}{r} = (2\cos(3t) - 1)\underset{\sim}{i} + \sin(3t)\underset{\sim}{j}$. Determine the Cartesian equation of the path.

$\underset{\sim}{r} = (2\cos(3t) - 1)\underset{\sim}{i} + \sin(3t)\underset{\sim}{j}$

$x = 2\cos(3t) - 1, \quad y = \sin(3t)$

$x + 1 = 2\cos(3t)$

$\cos(3t) = \dfrac{x+1}{2} \qquad \sin(3t) = \dfrac{y}{1}$

$\cos^2(3t) + \sin^2(3t) = 1$

$\dfrac{(x+1)^2}{4} + \dfrac{y^2}{1} = 1$

An ellipse with centre $(^{-}1, 0)$, $a = 2$, $b = 1$

Review Questions

1. Find the Cartesian equation for the path of an object with the position vector

a. $\underset{\sim}{r}(t) = \ln(2t)\underset{\sim}{i} + (4t - 1)\underset{\sim}{j}$.

b. $\underset{\sim}{r}(t) = (3e^{-2t} + 1)\underset{\sim}{i} + (2e^{-3t} - 1)\underset{\sim}{j}$

c. $\underset{\sim}{r} = (2\cos(3t) - 1)\underset{\sim}{i} + \sin(3t)\underset{\sim}{j}$

2. Find the distance of the body described by the following position vectors from the origin at the times listed, using distance = $\sqrt{\underset{\sim}{r}.\underset{\sim}{r}} = |\underset{\sim}{r}|$.

a. $\underset{\sim}{r}(t) = 2t^2\underset{\sim}{i} + \log(10t)\underset{\sim}{j}$ at $t = 1$

b. $\underset{\sim}{r}(t) = 4e^{-2t}\underset{\sim}{i} - 2t^{-1}\underset{\sim}{j}$ at $t = \frac{1}{2}$

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Solutions to Review Questions

1.

a.

$$x = \ln(2t), \quad e^x = 2t, \quad t = \frac{e^x}{2}$$

$$\begin{aligned} y(x) &= 4t - 1 \\ &= 4\left(\frac{e^x}{2}\right) - 1 \\ &= 2e^x - 1 \end{aligned}$$

b. Let $x = 3e^{-2t} + 1$

$$\begin{aligned} x - 1 &= 3e^{-2t} \\ e^{-2t} &= \frac{x-1}{3} \\ \left(e^{-2t}\right)^{\frac{3}{2}} &= \left(\frac{x-1}{3}\right)^{\frac{3}{2}} \\ e^{-3t} &= \left(\frac{x-1}{3}\right)^{\frac{3}{2}} \\ y &= 2e^{-3t} - 1 \\ &= 2\left(\frac{x-1}{3}\right)^{\frac{3}{2}} - 1 \end{aligned}$$

c. $\underset{\sim}{r} = (2\cos(3t) - 1)\,\underset{\sim}{i} + \sin(3t)\,\underset{\sim}{j}$

$x = 2\cos(3t) - 1, \quad y = \sin(3t)$

$x + 1 = 2\cos(3t)$

$\cos(3t) = \dfrac{x+1}{2} \qquad \sin(3t) = \dfrac{y}{1}$

$\cos^2(3t) + \sin^2(3t) = 1$

$$\frac{(x+1)^2}{4} + \frac{y^2}{1} = 1$$

2.

a. $|\underset{\sim}{r}| = \sqrt{\underset{\sim}{r}.\underset{\sim}{r}}$

$= \sqrt{2^2 + (\log(10))^2}$

$= \sqrt{4+1}$

$= \sqrt{5}$

b. $|\underset{\sim}{r}| = \sqrt{(4e^{-1})^2 + \left({}^{-}2 \times \left(\frac{1}{2}\right)^{-1}\right)^2}$

$= \sqrt{\frac{16}{e^2} + ({}^{-}4)^2}$

$= \frac{1}{e}\sqrt{16 + 16e^2}$

3.

a. $\underset{\sim}{v}(t) = \frac{\mathrm{d}\,\underset{\sim}{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[(3t^2 - \sin(\pi t))\,\underset{\sim}{i} + (e^{-t} + 2t)\,\underset{\sim}{j}]$

$= (6t - \pi\cos(\pi t))\,\underset{\sim}{i} + ({}^{-}e^{-t} + 2)\,\underset{\sim}{j}$

b. $\underset{\sim}{v}(2) = (6 \times 2 - \pi\cos(2\pi))\,\underset{\sim}{i} + ({}^{-}e^{-2} + 2)\,\underset{\sim}{j}$

$= (12 - \pi)\,\underset{\sim}{i} + (2 - e^{-2})\,\underset{\sim}{j}$

c. Average velocity $= \frac{\underset{\sim}{r}(2) - \underset{\sim}{r}(0)}{2}$

$= \frac{(3 \times 2^2)\,\underset{\sim}{i} + (e^{-2} - 1 + 2 \times 2)\,\underset{\sim}{j}}{2}$

$= 6\,\underset{\sim}{i} + \frac{e^{-2}+3}{2}j$

4.

a. $\underset{\sim}{r}(0) = (3e^{-2(0)} + 1)\,\underset{\sim}{i} + (2e^{-3(0)} - 1)\,\underset{\sim}{j} = 4\,\underset{\sim}{i} + \underset{\sim}{j}$

$\underset{\sim}{v}(t) = ({}^{-}6e^{-2t})\,\underset{\sim}{i} + ({}^{-}6e^{-3t})\,\underset{\sim}{j}$

$\underset{\sim}{v}(0) = ({}^{-}6e^{-2(0)})\,\underset{\sim}{i} + ({}^{-}6e^{-3(0)})\,\underset{\sim}{j} = {}^{-}6\,\underset{\sim}{i} - 6\,\underset{\sim}{j}$

$\underset{\sim}{a}(t) = (12e^{-2t})\,\underset{\sim}{i} + (18e^{-3t})\,\underset{\sim}{j}$

$\underset{\sim}{a}(0) = 12\,\underset{\sim}{i} + 18\,\underset{\sim}{j}$

b.